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Koszul Complexes and Hyperdeterminants

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Let

$$A = \sum_{1 \le j_i \le d_i} a_{j_1 \cdots j_r} x_{j_1}^{(1)} \cdots x_{j_r}^{(r)}$$

be an r-linear form in the polynomial ring $\overline{R} = \mathbf{C}[x_1^{(1)}, \ldots, x_{d_1}^{(1)}; \ldots; x_1^{(r)}, \ldots, x_{d_r}^{(r)}]$, $r \ge 2$, and $d_i \ge 2$ for every $i = 1, \ldots, r$.

Provided that

$$d_{i_0} - 1 \le \sum_{i \ne i_0} (d_i - 1) \qquad \forall i_0 \in \{1, \ldots, r\}, \tag{1}$$

it makes sense to speak of the hyperdeterminant $\mathrm{Det}(A)$. It is an integer polynomial in the coefficients $a_{j_1 \cdots j_r}$, which is an irreducible equation for the dual variety $X^\vee$ of the product of projective spaces

$$X = \mathbf{P}^{d_1 - 1} \times \cdots \times \mathbf{P}^{d_r - 1}$$

embedded into $\mathbf{P}^{d_1 \cdots d_r - 1}$ by a multiple Segre embedding. (If one thinks of the r-dimensional matrix $(a_{j_1 \cdots j_r})$ of format $d_1 \times \cdots \times d_r$, Condition 1

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can be viewed as the multidimensional version of the prescription that an ordinary, two-dimensional matrix be square.)

Hyperdeterminants were introduced and investigated in [5]. In that paper, the Cayley–Koszul complexes $C^{\cdot}(m_1, \cdots, m_r; A)$ also were introduced, where A is as above, and the m_i are integer parameters. Those are complexes of $\mathbf{C}$-vector spaces, whose boundary maps depend polynomially on the coefficients of A. It was then proved that, assuming Condition 1 holds, for every choice of *nonnegative* parameters m_i, $\mathrm{Det}(A)$ is equal to

$$[d(m_1, \ldots, m_r; A)]^{(-1)^{d_1+\cdots+d_r-r+1}},$$

where $d(m_1, \ldots, m_r; A)$ stands for the determinant of $C^{\cdot}(m_1, \ldots, m_r; A)$. (Since the determinant of a complex is defined up to a nonzero constant, the equality is understood to mean that $\mathrm{Det}(A)$ coincides with one of the possible expressions of $[d(m_1, \ldots, m_r; A)]^{(-1)^{d_1+\cdots+d_r-r+1}}$.)

The Cayley–Koszul complexes are still hard to work out explicitly. In this paper, we recover all those having $m_i > 0$ for every i, as multigraded components of an appropriate Koszul complex $\mathbf{K}_A$ on a certain non-free $\overline{R}$-module M, introduced in [8]. The construction is very explicit and uses only some basic tools of algebra. (The case $r = 2$ was already in [2].)

Our approach allows us also to interpret $\mathrm{Det}(A)$ in terms of the determinant of a multigraded component of a double complex $\mathbf{D}_A$ of free $\overline{R}$-modules.

The organization of this paper is as follows. In Section 1, the complexes $\mathbf{K}_A$ and $\mathbf{D}_A$ are defined over any Noetherian commutative ring (the Noetherian assumption could in fact be removed). In Section 2, we calculate the multigraded components of the exterior powers $\Lambda^v M$, in terms of representation theory: the ground ring is now supposed to be a characteristic zero field (but it would be enough that it included a copy of $\mathbf{Q}$). Section 3 contains the main results, under the assumption that the ground ring is an algebraically closed field of characteristic zero, essentially $\mathbf{C}$; it contains also something on the Jacobian ideal of the given form A.

Throughout what follows, whenever we speak of $\mathrm{Det}(A)$, it is implicitly assumed that Condition 1 is in force.

1. DEFINITIONS AND FIRST PROPERTIES

Let R be any Noetherian commutative ring.

Let $F_1, \ldots, F_r$ ($r \geq 2$) denote r free R-modules of ranks $d_1, \ldots, d_r$, resp. (each $d_i \geq 2$). For every $i = 1, \ldots, r$, we choose a basis $\left\{f_1^{(i)}, \ldots, f_{d_i}^{(i)}\right\}$ for

F_i and denote by $\left\{x_1^{(i)}, \ldots, x_{d_i}^{(i)}\right\}$ its dual basis. If we let $\overline{R}$ stand for the symmetric algebra $S(F_1^* \oplus \cdots \oplus F_r^*) \cong R\left[x_1^{(1)}, \ldots, x_{d_1}^{(1)}; \ldots; x_1^{(r)}, \ldots, x_{d_r}^{(r)}\right]$, then we obtain r free $\overline{R}$-modules by extension of scalars: $\overline{F}_i = F_i \otimes_R \overline{R}$, for every i.

The basis of $\overline{F}_i$ induced by $\left\{f_1^{(i)}, \ldots, f_{d_i}^{(i)}\right\}$ will still be indicated by the same symbols. The basis of $\overline{F}_i^*$ induced by $\left\{x_1^{(i)}, \ldots, x_{d_i}^{(i)}\right\}$ will be indicated similarly.

Let $\psi : \underbrace{\overline{R} \oplus \cdots \oplus \overline{R}}_{r-1} \to \overline{F}_1 \oplus \cdots \oplus \overline{F}_r$ be the map defined by

$$\psi(0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th place}}}{1}, 0, \ldots, 0) = \sum_{1 \le j \le d_i} x_j^{(i)} f_j^{(i)} + \sum_{1 \le k \le d_{i+1}} x_k^{(i+1)} f_k^{(i+1)},$$

where i ranges in $\{1, \ldots, r-1\}$. The (row-wise) matrix associated to ψ is

$$U = \begin{pmatrix} \underline{x}^{(1)} & \underline{x}^{(2)} & \underline{0} & \underline{0} \ldots \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{x}^{(2)} & \underline{x}^{(3)} & \underline{0} \ldots \underline{0} & \underline{0} & \underline{0} \\ \vdots & \vdots & \vdots & \vdots \quad\;\; \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} \ldots \underline{0} & \underline{x}^{(r-1)} & \underline{x}^{(r)} \end{pmatrix},$$

where each $\underline{x}^{(i)}$ is the row vector $(x_1^{(i)}, \ldots, x_{d_i}^{(i)})$ and each $\underline{0}$ stands for a zero row vector of appropriate length.

Given any r-dimensional matrix $A = (a_{j_1 \cdots j_r})$ of format $d_1 \times \cdots \times d_r$ with entries in R, we identify it with the multilinear form

$$\sum_{1 \le j_i \le d_i} a_{j_1 \cdots j_r} x_{j_1}^{(1)} \cdots x_{j_r}^{(r)},$$

an element of $\overline{R}$. It makes sense to consider the partial derivatives of A and define a map φ_A: $\overline{F}_1 \oplus \cdots \oplus \overline{F}_r \to \overline{R}$ by means of

$$\varphi_A(f_j^{(i)}) = (-1)^{i+1} \frac{\partial A}{\partial x_j^{(i)}}.$$

Clearly, $\varphi_A \circ \psi = 0$, and φ_A induces a morphism $\overline{\varphi}_A$: $\mathrm{Coker}(\psi) \to \overline{R}$, which depends on A.

Definition 1. For every A, $\mathbf{K}_A$ is the Koszul complex

$$0 \to \Lambda^\ell M \overset{\partial_A^\ell}{\to} \Lambda^{\ell-1} M \overset{\partial_A^{\ell-1}}{\to} \cdots \overset{\partial_A^3}{\to} \Lambda^2 M \overset{\partial_A^2}{\to} M \overset{\overline{\varphi}_A}{\to} \overline{R},$$

where M stands for $\mathrm{Coker}(\psi)$ and $\ell = d_1 + \cdots + d_r$.

The finitely generated $\overline{R}$-module M (independent of A) is *not* free over $\overline{R}$. For every $v = 1, \ldots, \ell$, $\Lambda^v M$ has the presentation

$$\left(\underbrace{\overline{R} \oplus \cdots \oplus \overline{R}}_{r-1}\right)$$
$$\otimes \Lambda^{v-1}\left(\overline{F}_1 \oplus \cdots \oplus \overline{F}_r\right) \xrightarrow{m_\Lambda \circ (\psi \otimes 1)} \Lambda^v\left(\overline{F}_1 \oplus \cdots \oplus \overline{F}_r\right) \longrightarrow \Lambda^v M \longrightarrow 0,$$

since $\Lambda^v M = H_0(\mathbf{L}_{(v)}\psi)$, where $\mathbf{L}_{(v)}\psi$ denotes the Schur complex associated to the map ψ and the partition (v) (cf. [1, Proposition V.2.2]).

For the convenience of the reader, we pause for a moment in order to recall the definition of the Schur complex given in [1]. Let R_0 be any commutative ring and ξ any homomorphism $G_0 \to F_0$ of finitely generated free modules over R_0. Think of ξ as a complex with F_0 in degree 0 and G_0 in degree 1. Taking the divided power algebra DG_0 and the exterior algebra ΛF_0, consider $DG_0 \otimes \Lambda F_0$ as an algebra with respect to

$$(a \otimes b)(a' \otimes b') = (-1)^{jk} aa' \otimes bb',$$

where $b \in \Lambda^j F_0$ and $a' \in D_k G_0$; $DG_0 \otimes \Lambda F_0$ is a Hopf algebra, which can be endowed with the structure of a complex by means of the map ξ: the degree j component of the complex is $\sum_{i\geq 0} D_j G_0 \otimes \Lambda^i F_0$ and the differential is the R_0-map given by the action of $c(\xi) \in G_0^* \otimes F_0$, the trace of ξ, on $DG_0 \otimes \Lambda F_0$. We denote $DG_0 \otimes \Lambda F_0$, thought of as a complex, by $\mathbf{\Lambda}\xi$. One can express $\mathbf{\Lambda}\xi$ as a direct sum of complexes, $\mathbf{\Lambda}\xi = \sum_{k\geq 0} \mathbf{\Lambda}^k \xi$, where $\Lambda^k \xi$ stands for

$$0 \to D_k G_0 \to \cdots \to D_i G_0 \otimes \Lambda^{k-i} F_0$$
$$\to D_{i-1} G_0 \otimes \Lambda^{k-i+1} F_0 \to \cdots \to \Lambda^k F_0.$$

Similarly, taking the symmetric algebra SF_0 and the Hopf algebra $\Lambda G_0 \otimes SF_0$, the action of $c(\xi)$ on $\Lambda G_0 \otimes SF_0$ defines a differential $\sum_{i\geq 0} \Lambda^j G_0 \otimes S_i F_0 \longrightarrow \sum_{i\geq 0} \Lambda^{j-1} G_0 \otimes S_i F_0$. Writing $\mathbf{S}\xi$ for the complex $\Lambda G_0 \otimes SF_0$, one gets $\mathbf{S}\xi = \sum_{k\geq 0} \mathbf{S}_k \xi$, where $\mathbf{S}_k \xi$ stands for

$$0 \to \Lambda^k G_0 \to \cdots \to \Lambda^i G_0 \otimes S_{k-i} F_0$$
$$\to \Lambda^{i-1} G_0 \otimes S_{k-i+1} F_0 \to \cdots \to S_k F_0.$$

Notice that the Hopf algebras $\mathbf{\Lambda}\xi$ and $\mathbf{S}\xi$ have diagonals and multiplications which are *compatible* with the complex structure. Hence the following are morphisms of complexes: the diagonals

$$\mathbf{\Lambda}^k \xi \xrightarrow{\Delta} \underbrace{\mathbf{\Lambda}^1 \xi \otimes \ldots \otimes \mathbf{\Lambda}^1 \xi}_{k}$$

and the multiplications

$$\underbrace{\mathbf{S}_1\xi \otimes \cdots \otimes \mathbf{S}_1\xi}_{k} \xrightarrow{m} \mathbf{S}_k\xi.$$

Given any partition $\lambda = (\ell_1, \ldots, \ell_s)$, and denoting its conjugate partition by $(\ell_1', \ldots, \ell_t')$, let $d_\lambda(\xi)$ indicate the composite map of complexes

$$\begin{array}{ccc}
\mathbf{\Lambda}^{\ell_1}\xi & \otimes\cdots\otimes & \mathbf{\Lambda}^{\ell_s}\xi \\
& \Big\downarrow \Delta^{\otimes s} & \\
\underbrace{\xi(1,1)\otimes\cdots\otimes\xi(1,\ell_1)}_{\ell_1 \text{ copies of } \xi} & \otimes\cdots\otimes & \underbrace{\xi(\ell_1',1)\otimes\cdots\otimes\xi(\ell_1',\ell_s)}_{\ell_s \text{ copies of } \xi} \\
& \Big\downarrow \text{perm} & \\
\underbrace{\xi(1,1)\otimes\cdots\otimes\xi(\ell_1',1)}_{\ell_1' \text{ copies of } \xi} & \otimes\cdots\otimes & \underbrace{\xi(1,\ell_s)\otimes\cdots\otimes\xi(\ell_t',\ell_s)}_{\ell_t' \text{ copies of } \xi} \\
& \Big\downarrow m^{\otimes t} & \\
\mathbf{S}_{\ell_1'}\xi & \otimes\cdots\otimes & \mathbf{S}_{\ell_t'}\xi,
\end{array}$$

where we have used the fact that $s = \ell_1'$ and $t = \ell_1$, as well as the fact that $\mathbf{\Lambda}^1\xi = \xi = \mathbf{S}_1\xi$.

By definition, the image of $d_\lambda(\xi)$ is the Schur complex associated to the map ξ and the partition λ, $\mathbf{L}_\lambda\xi$. In particular, $\mathbf{L}_\lambda\xi = \mathbf{\Lambda}^t\xi$ when $\lambda = (t)$ and $\mathbf{L}_\lambda\xi = \mathbf{S}_s\xi$ when $\lambda = (1^s)$. If λ is any, but $G_0 = 0$, then $\mathbf{L}_\lambda\xi$ is concentrated in degree 0: its unique nonzero term is called the Schur module associated to F_0 and λ, $L_\lambda F_0$. $L_\lambda F_0$ is a representation of the general linear group $GL(F_0)$; whenever R_0 is a characteristic zero field, $GL(F_0)$ is linearly reductive, and for every d, $\{L_\lambda F_0 : \lambda \text{ is a partition of } d\}$ is a complete set of degree d homogeneous irreducible representations of $GL(F_0)$.

Going back to $\Lambda^v M = H_0(\mathbf{L}_{(v)}\psi)$, in fact $\mathbf{L}_{(v)}\psi$ may or may not be a resolution of $\Lambda^v M$, as the following proposition indicates.

PROPOSITION 2. $\mathbf{L}_{(v)}\psi$ *is a resolution of* $\Lambda^v M$, *whenever*

$$1 \le v \le \min_{1\le i_1<i_2\le r}\left\{d_{i_1} + d_{i_2}\right\}.$$

Proof. We resort to the Acyclicity Lemma, as stated for instance in [4, Corollary 1.3].

Since $\mathbf{L}_{(v)}\psi$ has length v, it suffices to show acyclicity after localization at primes P such that depth $P\overline{R}_P < v$. We claim that, after localizing at such a prime P, the matrix U turns out to have an invertible element in every row, no two such invertible elements belonging to the same column. Thus by means of a suitable change of basis, we can assume that ψ_P has the matrix

$$\begin{pmatrix} \underline{a}^{(1)} & \underline{0} & \underline{0}\,\underline{0}\ldots\underline{0}\,\underline{0} & \underline{0} \\ \underline{0} & \underline{a}^{(2)} & \underline{0}\,\underline{0}\ldots\underline{0}\,\underline{0} & \underline{0} \\ \vdots & \vdots & \vdots\;\vdots\quad\;\vdots\;\vdots & \vdots \\ \underline{0} & \underline{0} & \underline{0}\,\underline{0}\ldots\underline{0}\,\underline{a}^{(r-1)} & \underline{0} \end{pmatrix},$$

where each $\underline{a}^{(i)}$ stands for the row vector $(1, 0, \ldots, 0)$ of length d_i and each $\underline{0}$ is a zero row vector of appropriate length. It follows that the cokernel of ψ_P is a free module (of rank $\ell - r + 1$), i.e., we have a split monomorphism, and [1, Corollary V.1.15] gives the required acyclicity.

Let us prove our claim. After localizing at the above mentioned prime P, one of the indeterminates, say $x_j^{(i)}$, becomes invertible, because

$$\operatorname{depth}\left(\underline{x}^{(1)}; \ldots; \underline{x}^{(r)}\right) = \ell \geq v.$$

If $r = 2$, we are done. Otherwise ($r \geq 3$),

$$\operatorname{depth}\left(\underline{x}^{(1)}; \ldots; \underline{x}^{(i-1)}; \underline{x}^{(i+1)}; \ldots; \underline{x}^{(r)}\right) = \ell - d_i \geq \min_{1 \leq d_1 < d_2 \leq r} \left\{d_{i_1} + d_{i_2}\right\} \geq v$$

tells us that some indeterminate $x_{j'}^{(i')}$, $i' \neq i$, becomes invertible, too.

If $r = 3$, the above concludes the proof. Otherwise ($r \geq 4$),

$$\begin{aligned} &\operatorname{depth}\left(\underline{x}^{(1)}; \ldots; \underline{x}^{(i-1)}; \underline{x}^{(i+1)}; \ldots; \underline{x}^{(i'-1)}; \underline{x}^{(i'+1)}; \ldots; \underline{x}^{(r)}\right) \\ &\qquad = \ell - (d_i + d_{i'}) \geq \min_{1 \leq d_1 < d_2 \leq r} \left\{d_{i_1} + d_{i_2}\right\} \geq v \end{aligned}$$

shows that a further indeterminate $x_{j''}^{(i'')}$, $i'' \neq i, i'$, becomes invertible, as well.

If $r = 4$, we are through. Otherwise ($r \geq 5$), we can continue finding other invertible elements in the remaining rows, and after a finite number of steps we end the proof. ∎

The complex $\mathbf{K}_A$ is closely related to the following double complex:

DEFINITION 3. For every A, $\mathbf{D}_A$ is the double complex

$$\begin{array}{ccccccccc}
0 \to & \Lambda^{\ell}\overline{E} & \to \dots \to & \Lambda^2\overline{E} & \to & \overline{E} & \to \overline{R} \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \to & \left(\Lambda^{\ell-1}\overline{E}\right)^{\oplus(r-1)} & \to \dots \to & \overline{E}^{\oplus(r-1)} & \to & \overline{R}^{\oplus(r-1)} & \\
& \uparrow & & \uparrow & & & \\
0 \to & \left(\Lambda^{\ell-2}\overline{E}\right)^{\oplus\frac{r(r-1)}{2}} & \to \dots \to & \overline{R}^{\oplus\frac{r(r-1)}{2}} & & & \\
& \uparrow & & & & & \\
& \vdots & & & & & \\
& \uparrow & & & & & \\
0 \to & \overline{R}^{\oplus\frac{(r+\ell-2)\dots r(r-1)}{\ell!}} & & & & &
\end{array}$$

where $\overline{E}$ stands for $\overline{F}_1 \oplus \cdots \oplus \overline{F}_r$, the columns are the complexes $\mathbf{L}_{(v)}\psi$ (recall that each divided power $D_u\left(\underbrace{\overline{R} \oplus \cdots \oplus \overline{R}}_{r-1}\right)$ has rank $\binom{(r-1)+u-1}{u}$), while the rows are provided by truncations of the Koszul complex over φ_A.

One should note that, in view of Proposition 2, the rightmost columns of $\mathbf{D}_A$ are acyclic and in the leftmost columns some homology appears.

PROPOSITION 4. (a) *For every $k = 1, \dots, r-1$, the ideal, $I_k(U)$, generated in $\overline{R}$ by the $k \times k$ minors of the matrix U coincides with the ideal generated by all products of type*

$$x_{j_{i_1}}^{(i_1)} \cdots x_{j_{i_k}}^{(i_k)}, \qquad 1 \le i_1 < \cdots < i_k \le r.$$

(b) *For every $k = 0, \dots, r-1$, the module $\Lambda^{\ell-k}M$ is an extension of $I_k(U)/I_{k+1}(U)$, where $I_0(U) = \overline{R}$ and $I_r(U) = (0)$. In particular, $\Lambda^{\ell}M = I_0(U)/I_1(U) = R$ and $\Lambda^{\ell-1}M = I_1(U)/I_2(U)$.*

Proof. For part (a), we consider two cases.

Case $k = r-1$ (Maximal Minors). If $r = 2$, the proof is trivial. So let us assume $r \ge 3$. A minor V is given by prescribing $r-1$ columns. If no two columns involve indeterminates from the same $\overline{F}_i^*$, V contains $r-1$ kinds of indeterminates (one kind in each column) and has the form

$$\begin{vmatrix} V_1 & 0 \\ 0 & V_2 \end{vmatrix},$$

where V_1 and V_2 are both triangular (one of them possibly empty); thus V equals the product of the terms on its diagonal, and such a product is precisely of the required type. If two columns of V involve indeterminates from the same $\overline{F}_i^*$, then V is of the kind

$$\begin{vmatrix} & 0 & 0 & \\ & \vdots & \vdots & \\ & 0 & 0 & \\ s \text{ initial} & x_j^{(i)} & x_{j'}^{(i)} & t \text{ final} \\ \text{columns} & x_j^{(i)} & x_{j'}^{(i)} & \text{columns} \\ & 0 & 0 & \\ & \vdots & \vdots & \\ & 0 & 0 & \end{vmatrix}, \qquad s+t=r-3;$$

by the generalized Laplace rule,

$$V = \begin{vmatrix} x_j^{(i)} & x_j^{(i)} \\ x_{j'}^{(i)} & x_{j'}^{(i)} \end{vmatrix} \cdot [(r-3)\text{–order determinant}],$$

and the first factor is zero.

Case $k \leq r-2$ (Non-maximal Minors). Every minor of order k equals a maximal minor of a matrix U' obtained from U by removing $r-1-k$ rows. U' is in fact a block matrix,

$$\begin{pmatrix} U_1' & & & \\ & U_2' & & \\ & & \ddots & \\ & & & U_{r-k}' \end{pmatrix},$$

where each block is a matrix U for a case with lower r and the indeterminates occurring in the block do not occur anywhere else in U'. But the only nonzero maximal minors of U' are products of maximal minors of the indicated blocks. Hence we are done, by the previous case.

Let us prove part (b). If we identify $\Lambda^{\ell-k}\overline{E}$ with $\Lambda^k\overline{E}^*$, $\Lambda^{\ell-k}M$ is the cokernel of the composite map:

$$\alpha : \overline{R}^{\oplus(r-1)} \otimes \Lambda^{k+1}\overline{E}^* \xrightarrow{\psi\otimes\Delta} \overline{E} \otimes \overline{E}^* \otimes \Lambda^k\overline{E}^* \xrightarrow{ev\otimes 1} \overline{R} \otimes \Lambda^k\overline{E}^* \cong \Lambda^k\overline{E}^*.$$

Let β: $\Lambda^k\overline{E}^* \to I_k(U)$ be the morphism sending each $x_{j_{i_1}}^{(i_1)} \wedge \cdots \wedge x_{j_{i_k}}^{(i_k)}$ (k different spaces $\overline{F}_i^*$ involved) to $x_{j_{i_1}}^{(i_1)} \cdots x_{j_{i_k}}^{(i_k)}$, and all the other elements of the canonical basis to 0. If π stands for the projection $I_k(U) \to I_k(U)/I_{k+1}(U)$, we claim that $\pi \circ \beta \circ \alpha = 0$, i.e., β induces a surjective map:

$$\Lambda^{\ell-k}M \longrightarrow I_k(U)/I_{k+1}(U).$$

Indeed, take an element of the canonical basis of $\Lambda^{k+1}\overline{E}^*$, say e. If e contains three (or more) factors coming from the same $\overline{F}_i^*$, $\beta \circ \alpha$ sends to 0 every product between e and an element of $\overline{R}^{\oplus(r-1)}$. If e contains two factors from some $\overline{F}_i^*$ and two factors from $\overline{F}_{i'}^*$, $i' \neq i$, again $\beta \circ \alpha$ sends to 0 every product between e and an element of $\overline{R}^{\oplus(r-1)}$. If e contains two factors from some $\overline{F}_i^*$ and $k-1$ further factors from *different* spaces, $(\beta \circ \alpha)(1_i \otimes e) = 0$ after some canceling, and $(\beta \circ \alpha)(1_{i'} \otimes e) = 0$ directly, for every $i' \neq i$. Here 1_i denotes the element of $\overline{R}^{\oplus(r-1)}$,

$$(0, \ \ldots, \ 0, \ \underset{\substack{\uparrow \\ i\text{th place}}}{1}, \ 0, \ \ldots, \ 0).$$

Finally, let us see what happens to $e = x_{j_{i_1}}^{(i_1)} \wedge \cdots \wedge x_{j_{i_{k+1}}}^{(i_{k+1})}$: for every 1_i, we get $\alpha(1_i \otimes e)$ equal to

$$\pm\Bigg(x_{j_{i+1}}^{(i+1)} \cdot x_{j_{i_1}}^{(i_1)} \wedge \cdots \wedge \widehat{x_{j_{i+1}}^{(i+1)}} \wedge \cdots \wedge x_{j_{i_{k+1}}}^{(i_{k+1})} - x_{j_i}^{(i)} \cdot x_{j_{i_1}}^{(i_1)} \wedge \cdots \wedge \widehat{x_{j_i}^{(i)}} \wedge \cdots \wedge x_{j_{k+1}}^{(i_{k+1})} \Bigg)$$

(the hat on an indeterminate means that it is omitted); if both $x_{j_i}^{(i)}$ and $x_{j_{i+1}}^{(i+1)}$ occur in e, β sends $\alpha(1_i \otimes e)$ to 0; if neither $x_{j_i}^{(i)}$ nor $x_{j_{i+1}}^{(i+1)}$ occurs in e, $\alpha(1_i \otimes e)$ is already 0; if just one of them occurs, β sends $\alpha(1_i \otimes e)$ to (say)

$$x_{j_{i+1}}^{(i+1)} \cdot \left(x_{j_{i_1}}^{(i_1)} \cdot \ldots \cdot \widehat{x_{j_{i+1}}^{(i+1)}} \cdot \ldots \cdot x_{j_{i_{k+1}}}^{(i_{k+1})} \right) = x_{j_{i_1}}^{(i_1)} \cdot \ldots \cdot x_{j_{i_{k+1}}}^{(i_{k+1})} \in I_{k+1}(U),$$

which is $\neq 0$, but is sent to 0 by π.

To complete part (b), it is enough to observe that when $k = 0, 1$, the map $\Lambda^{\ell-k}M \to I_k(U)/I_{k+1}(U)$ induced by β is injective, too. ∎

Remark 1. Using the description of $\Lambda^\ell M$ and $\Lambda^{\ell-1}M$ given in the proposition above, it is easy to check that the map ∂_A^ℓ of $\mathbf{K}_A$ is always 0, which implies $H_\ell(\mathbf{K}_A) = R$, and that if $r \geq 3$, the map $\partial_A^{\ell-1}$ is 0 as well, which implies $H_{\ell-1}(\mathbf{K}_A) = I_1(U)/I_2(U)$. We shall prove in Section 2 that in fact all maps $\partial_A^{\ell-k}$ of $\mathbf{K}_A$ are 0, when $k = 0, \ldots, r-2$, so that $H_{\ell-k}(\mathbf{K}_A) = \Lambda^{\ell-k}M$ for every $k = 0, \ldots, r-2$.

We end this section by pointing out that the group

$$G = GL(F_1^*) \times \cdots \times GL(F_r^*)$$

acts naturally on both $\mathbf{K}_A$ and $\mathbf{D}_A$.

2. THE DECOMPOSITION OF $\Lambda^v M$

In this section we assume that the ground ring R is a characteristic zero field.

It follows that $GL(F_i^*)$ is linearly reductive for every i, and each representation of G decomposes as a direct sum of irreducibles. The irreducible G-modules are all products,

$$L_{\lambda_1} F_1^* \otimes \cdots \otimes L_{\lambda_r} F_r^*,$$

where $L_{\lambda_i} F_i^*$ is a Schur module for the group $GL(F_i^*)$. (The definition of Schur module has been recalled after that of Schur complex. Also cf. [1].)

We wish to decompose into irreducibles every $\Lambda^v M$, $0 \leq v \leq \ell$. It is enough to give the decomposition of each multigraded component of $\Lambda^v M$.

Let us recall the double complex $\mathbf{D}_A$ of Definition 3, and denote by $\mathbf{D}_A(u, v)$ the term occurring at the intersection of the vth column (the one with $\Lambda^v \overline{E}$ on top) and the uth row ($u \geq 0$ *increasing* from top to bottom). The multigraded component $(\mathbf{D}_A(u, v))_{\underline{a}}$, $\underline{a} = (a_1, \ldots, a_r)$ with each a_i a nonnegative integer, coincides with

$$\left[\sum_{\alpha_1 + \cdots + \alpha_r = v-u} (\Lambda^{\alpha_1} F_1 \otimes S_{a_1+\alpha_1-v} F_1^*) \otimes \cdots \otimes (\Lambda^{\alpha_r} F_r \otimes S_{a_r+\alpha_r-v} F_r^*)\right]^{\oplus \binom{(r-1)+u-1}{u}}$$

$$\cong \left[\sum_{\alpha_1 + \cdots + \alpha_r = v-u} (\Lambda^{d_1-\alpha_1} F_1^* \otimes S_{a_1+\alpha_1-v} F_1^*) \otimes \cdots \otimes (\Lambda^{d_r-\alpha_r} F_r^* \otimes S_{a_r+\alpha_r-v} F_r^*)\right]^{\oplus \binom{(r-1)+u-1}{u}}$$

Hence the multigraded component $(\Lambda^v M)_{\underline{a}}$ is the cokernel of the morphism

$$\sum_{\alpha_1 + \cdots + \alpha_r = v} (\Lambda^{d_1-\alpha_1} F_1^* \otimes S_{a_1+\alpha_1-v} F_1^*) \otimes \cdots \otimes (\Lambda^{d_r-\alpha_r} F_r^* \otimes S_{a_r+\alpha_r-v} F_r^*)$$

$$\uparrow \ \vartheta_{\underline{a}}$$

$$\left[\sum_{\beta_1 + \cdots + \beta_r = v-1} (\Lambda^{d_1-\beta_1} F_1^* \otimes S_{a_1+\beta_1-v} F_1^*) \otimes \cdots \otimes (\Lambda^{d_r-\beta_r} F_r^* \otimes S_{a_r+\beta_r-v} F_r^*)\right]^{\oplus (r-1)}$$

which is the appropriate multigraded component of the map $m_\Lambda \circ (\psi \otimes 1)$ mentioned immediately before Proposition 2. Explicitly, on every summand of its domain, say the ith summand ($1 \le i \le r-1$), $\vartheta_{\underline{a}}$ acts as

$$1 \otimes \cdots \otimes 1 \otimes \underset{\substack{\uparrow \\ i\text{th place}}}{\vartheta_{\underline{a}}^{(i)}} \otimes 1 \otimes \cdots \otimes 1$$

$$+ 1 \otimes \cdots \otimes 1 \otimes \underset{\substack{\uparrow \\ (i+1)\text{st place}}}{\vartheta_{\underline{a}}^{(i+1)}} \otimes 1 \otimes \cdots \otimes 1$$

on each one of the terms parametrized by $(\beta_1, \ldots, \beta_r)$; here,

$$\vartheta_{\underline{a}}^{(i)} : \Lambda^{d_i - \beta_i} F_i^* \otimes S_{a_i + \beta_i - v} F_i^* \longrightarrow_{m_S \circ \Delta_\Lambda} \Lambda^{d_i - \beta_i - 1} F_i^* \otimes S_{a_i + \beta_i + 1 - v} F_i^*$$

and

$$\vartheta_{\underline{a}}^{(i+1)} : \Lambda^{d_{i+1} - \beta_{i+1}} F_{i+1}^* \otimes S_{a_{i+1} + \beta_{i+1} - v} F_{i+1}^* \longrightarrow_{m_S \circ \Delta_\Lambda} \Lambda^{d_{i+1} - \beta_{i+1} - 1} F_{i+1}^* \otimes S_{a_{i+1} + \beta_{i+1} + 1 - v} F_{i+1}^*$$

are the usual boundary maps of the Schur complexes $\mathbf{L}_{(1^{d_i + a_i - v})}(F_i^* \xrightarrow{\text{id}} F_i^*)$ and $\mathbf{L}_{(1^{d_{i+1} + a_{i+1} - v})}(F_{i+1}^* \xrightarrow{\text{id}} F_{i+1}^*)$, respectively. (For Schur complexes, again cf. [1].)

The irreducible components of

$$\sum_{\alpha_1 + \cdots + \alpha_r = v} (\Lambda^{d_1 - \alpha_1} F_1^* \otimes S_{a_1 + \alpha_1 - v} F_1^*) \otimes \cdots \otimes (\Lambda^{d_r - \alpha_r} F_r^* \otimes S_{a_r + \alpha_r - v} F_r^*),$$

which are not covered by $\mathrm{Im}(\vartheta_{\underline{a}})$, provide the required irreducible decomposition of $(\Lambda^v M)_{\underline{a}}$.

THEOREM 5. *With notation as above, it turns out that*

$$(\Lambda^v M)_{\underline{a}} = \sum_{\alpha_1 + \cdots + \alpha_r = v} L_{(d_1 - \alpha_1,\, 1^{a_1 + \alpha_1 - v})} F_1^* \otimes \cdots \otimes L_{(d_r - \alpha_r,\, 1^{a_r + \alpha_r - v})} F_r^*$$

$$+ \sum_{\gamma_1 + \cdots + \gamma_r = v-1,\ \gamma_i \neq d_i\ \forall i} L_{(d_1 - \gamma_1,\, 1^{a_1 + \gamma_1 - v})} F_1^* \otimes \cdots \otimes L_{(d_r - \gamma_r,\, 1^{a_r + \gamma_r - v})} F_r^*.$$

(*By the definition of Schur module, in both summations above a factor* $L_{(s,\, 1^t)} F^*$ *is automatically* 0, *if* $s < 0$, *or* $t < 0$, *or* $s = 0$ *together with* $t \ge 1$.)

Proof. It is well known that $\Lambda^{d_i - \alpha_i} F_i^* \otimes S_{a_i + \alpha_i - v} F_i^*$ decomposes as a $GL(F_i^*)$-module into the sum of two *hook* Schur modules:

$$L_{(d_i - \alpha_i + 1,\, 1^{a_i + \alpha_i - v - 1})} F_i^* \ \oplus\ L_{(d_i - \alpha_i,\, 1^{a_i + \alpha_i - v})} F_i^*.$$

We denote the first summand by $S_0(d_i - \alpha_i; a_i; v)$, and the second one by $S_1(d_i - \alpha_i; a_i; v)$. Notice the following:

(i) If $a_i + \alpha_i - v = 0$, then $S_0(d_i - \alpha_i; a_i; v) = 0$; in particular, if also $d_i - \alpha_i = 0$, then $\Lambda^{d_i-\alpha_i} F_i^* \otimes S_{a_i+\alpha_i-v} F_i^* = S_1(d_i - \alpha_i; a_i; v)$ is isomorphic to R.

(ii) If $d_i - \alpha_i = 0$ and $a_i + \alpha_i - v \geq 1$, then $S_1(d_i - \alpha_i; a_i; v) = 0$.

The whole $(\mathbf{D}_A(0, v))_{\underline{a}}$ decomposes as a G-module into a sum of tensor products of r hook Schur modules, each factor belonging to a different $GL(F_i^*)$. We denote by

$$S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$$

the tensor product

$$S_{k_1}(d_1 - \alpha_1; a_1; v) \otimes \cdots \otimes S_{k_r}(d_r - \alpha_r; a_r; v), \qquad k_i \in \{0, 1\}.$$

Remark that in order to have $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v) \neq 0$, it is necessary that:

(i) $k_i = 1$ for every i such that $a_i + \alpha_i - v = 0$

(ii) $k_i = 0$ for every i such that $d_i - \alpha_i = 0$ and $a_i + \alpha_i - v \geq 1$.

Also remark that a factor $S_{k_i}(d_i - \alpha_i; a_i; v)$ can be isomorphic to R if, and only if, $k_i = 1$ and $d_i - \alpha_i = 0 = a_i + \alpha_i - v$.

Given a G-irreducible $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$, its weight, w, is defined as the number of indices k_i such that $k_i = 1$ and $d_i - \alpha_i \neq 0$. Clearly, $w \leq |k| \leq r$, where $|k| = k_1 + \cdots + k_r$.

CLAIM. (a) *When either* $|k| \leq r - 2$, *or* $|k| = r - 1$ *and* $w < |k|$, *then*

$$S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v) \subseteq \operatorname{Im}(\vartheta_{\underline{a}}).$$

(b) *When either* $|k| = r - 1 = w$, *or* $|k| = r$, *then* $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ *occurs in* $\operatorname{Coker}(\vartheta_{\underline{a}})$ *with multiplicity* 1.

Proof of the Claim. Given any $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ of fixed weight w, such that $w \neq 0$ and $|k| \neq r$, there are in $(\mathbf{D}_A(0, v))_{\underline{a}}$ further

$$\sum_{h=1}^{\min\{w, r-|k|\}} \binom{w}{h} \binom{r - |k|}{h}$$

terms $S_{k'_1,\ldots,k'_r}(d_1 - \alpha'_1, \ldots, d_r - \alpha'_r; \underline{a}; v)$ isomorphic to it, and with equal weight, because:

$$S_0(d_{i_1} - \alpha_{i_1}; a_{i_1}; v) \otimes S_1(d_{i_2} - \alpha_{i_2}; a_{i_2}; v)$$

$$\cong S_1(d_{i_1} - (\alpha_{i_1} - 1); a_{i_1}; v) \otimes S_0(d_{i_2} - (\alpha_{i_2} + 1); a_{i_2}; v).$$

Whenever either $w = 0$, or $|k| = r$, then $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ occurs in $(\mathbf{D}_A(0, v))_{\underline{a}}$ with multiplicity 1.

Now, if $|k| = r$, (the only copy of) $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ cannot be covered by any term of $(\mathbf{D}_A(1, v))_{\underline{a}}$, since in our current notation, the domain of $\vartheta_{\underline{a}}^{(i)}$ is $S_0(d_i - \beta_i; a_i; v) \oplus S_1(d_i - \beta_i; a_i; v)$, and $\vartheta_{\underline{a}}^{(i)}(S_0(d_i - \beta_i; a_i; v)) = 0$, by the definition of hook Schur module, while

$$\vartheta_{\underline{a}}^{(i)}(S_1(d_i - \beta_i; a_i; v)) = \begin{cases} S_0(d_i - (\beta_i + 1); a_i; v) & \text{if} d_i - \beta_i \geq 1 \\ 0 & \text{if} d_i - \beta_i = 0. \end{cases}$$

If $|k| = r - 1$, we have in $(\mathbf{D}_A(0, v))_{\underline{a}}$ exactly $w + 1$ copies of $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$. Let us stipulate that in the string $k_1, \ldots, k_r$, it is $k_{i_0} = 0$ and $k_i = 1$ for all $i \neq i_0$. For every $i = 1, \ldots, r$, take in the ith summand of $(\mathbf{D}_A(1, v))_{\underline{a}}$ the G-irreducible term

$$S_{1,\ldots,1}(d_1 - \alpha_1, \ldots, d_{i_0-1} - \alpha_{i_0-1}, d_{i_0} - \alpha_{i_0} + 1, d_{i_0+1} - \alpha_{i_0+1}, \ldots, d_r - \alpha_r; \underline{a}; v).$$

Then the map $\vartheta_{\underline{a}}$ yields precisely $w + 1 - \delta_{|k|}^{w}$ copies of $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$, where $\delta_{|k|}^{w}$ is a Kronecker delta. It follows that $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ occurs in $\mathrm{Coker}(\vartheta_{\underline{a}})$ (with multiplicity 1) only when $w = |k|$.

If $|k| \leq r - 2$, we find at least two 0 in the string $k_1, \ldots, k_r$; let us call i_0 the smallest i for which $k_i = 0$, and i_1 the index such that $k_{i_1} = 0$ and $k_i = 1$ for every i between i_0 and i_1. We set $t = i_1 - i_0$. Then

$$\vartheta_{\underline{a}}^{(i_0)}[S_{k_1,\ldots,k_{i_0-1},k_{i_0}+1,k_{i_0+1},\ldots,k_r}(d_1 - \alpha_1, \ldots, d_{i_0-1} - \alpha_{i_0-1}, d_{i_0} - \alpha_{i_0} + 1, d_{i_0+1} - \alpha_{i_0+1}, \ldots, d_r - \alpha_r; \underline{a}; v)]$$

is precisely isomorphic to $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$, if $t = 1$; otherwise, it is isomorphic to $S_{k_1,\ldots,k_r}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ plus a term with lower t, and a suitable recursive argument concludes the proof of the Claim.

By means of the Claim, we finally prove the theorem. The irreducible G-modules occurring in $(\Lambda^v M)_{\underline{a}}$ are of two kinds:

(1) $S_{1,\ldots,1}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ with $\alpha_1 + \cdots + \alpha_r = v$

(2) $S_{1,\ldots,1,0,1,\ldots,1}(d_1 - \alpha_1, \ldots, d_r - \alpha_r; \underline{a}; v)$ with $\alpha_1 + \cdots + \alpha_r = v$ and the further requirement that $d_i - \alpha_i \neq 0$ for every $i \neq i_0$ (i_0 is the place where 0 occurs in the string $1, \ldots, 1, 0, 1, \ldots, 1$).

The first kind gives

$$L_{(d_1-\alpha_1,\,1^{a_1+\alpha_1-v})}F_1^* \otimes \cdots \otimes L_{(d_r-\alpha_r,\,1^{a_r+\alpha_r-v})}F_r^*, \qquad \alpha_1+\cdots+\alpha_r = v$$

(and some factors may be isomorphic to R). The second kind gives a product of factors all different from R:

$$\begin{aligned} &L_{(d_1-\alpha_1,\,1^{a_1+\alpha_1-v})}F_1^* \\ &\quad \otimes \cdots \otimes L_{(d_{i_0-1}-\alpha_{i_0-1},\,1^{a_{i_0-1}+\alpha_{i_0-1}-v})}F_{i_0-1}^* \otimes L_{(d_{i_0}-\alpha_{i_0}+1,\,1^{a_{i_0}+\alpha_{i_0}-1-v})}F_{i_0}^* \\ &\quad \otimes L_{(d_{i_0+1}-\alpha_{i_0+1},\,1^{a_{i_0+1}+\alpha_{i_0+1}-v})}F_{i_0+1}^* \otimes \cdots \otimes L_{(d_r-\alpha_r,\,1^{a_r+\alpha_r-v})}F_r^* \\ &\quad \cong L_{(d_1-\gamma_1,\,1^{a_1+\gamma_1-v})}F_1^* \otimes \cdots \otimes L_{(d_r-\gamma_r,\,1^{a_r+\alpha_r-v})}F_r^*, \end{aligned}$$

where $\gamma_{i_0} = \alpha_{i_0} - 1$, $\gamma_i = \alpha_i$ for each $i \neq i_0$, and $\gamma_1 + \cdots + \gamma_r = \alpha_1 + \cdots + \alpha_r - 1 = v - 1$. ∎

Remark 2. (a) The condition $\gamma_i \neq d_i\ \forall i$ in the statement of Theorem 5 implies that if $v \geq \ell - r + 2$, then only the summation $\sum_{\alpha_1+\cdots+\alpha_r=v}$ occurs. For $v \geq \ell - r + 2$ implies $\gamma_1 + \cdots + \gamma_r = v - 1 \geq \ell - r + 1 = (d_1 + \cdots + d_r) - (r-1)$, and at least one γ_i must equal d_i.

(b) It follows from (a) that if $v \geq \ell - r + 2$, then every G-irreducible component of $\Lambda^v M$ contains a hook $\cong R$. This is no longer true for $v = \ell - r + 1$, since $\gamma_i = d_i - 1$ for every i yields the extra components $S_{c_1}F_1^* \otimes \cdots \otimes S_{c_r}F_r^*$, where $c_i - 1 = a_i + (d_i - 1) - (\ell - r + 1) \geq 0$ for every i.

(c) It is easy to derive from Theorem 5 another proof of Proposition 4 (b).

COROLLARY 6. *Let* $k = 0, \ldots, r-2$.

(a) $(\Lambda^{\ell-k}M)_{\underline{a}} \neq 0$ *implies* $a_i - (\ell - d_i) \leq 0$ *for some* i.

(b) *The map* $\partial_A^{\ell-k}$ *of* $\mathbf{K}_A$ *is* 0. *Hence* $H_{\ell-k}(\mathbf{K}_A) = \Lambda^{\ell-k}M$. (*Cf. Remark* 1.)

Proof. (a) By Remark 2 (a), the G-irreducible components of $(\Lambda^v M)_{\underline{a}}$, $v = \ell - k$, are given only by $\alpha_1 + \cdots + \alpha_r = v$. Since $\alpha_1 + \ldots + \alpha_r = \sum_{i=1}^r d_i - k$ implies that $\alpha_i = d_i$ for at least $r-k$ indices i, it follows that the relevant multiindices $\underline{a}$ must contain at least $r-k$ terms a_i such that $a_i = (\ell - d_i) - k$, with all the others satisfying $a_i > (\ell - d_i) - k$. But $a_i = (\ell - d_i) - k$ says that $a_i - (\ell - d_i) \leq 0$, because $r \geq 2$ and $0 \leq k \leq r-2$.

(b) Having just described when $(\Lambda^v M)_{\underline{a}} \neq 0$, let us turn to $\Lambda^{v-1}M$. As long as $\ell - k - 1 \geq \ell - r + 2$, the proof of (a) shows that $\Lambda^{v-1}M$ occurs in multiindices $\underline{a}$ such that $a_i = (\ell - d_i) - k - 1$ for at least $r-k-1$ indices i. So no G-irreducible component occurring in $\Lambda^v M$ can have its multiindex in common with any component of $\Lambda^{v-1}M$, and this means that $\partial_A^{\ell-k} = 0$ for every $k = 0, \ldots, r-3$.

In order to prove that $\partial_A^{\ell-r+2} = 0$, by the same argument above, it suffices to show that no G-irreducible component in $\Lambda^{\ell-r+2}M$ can be isomorphic to any one of the extra components of $\Lambda^{\ell-r+1}M$ mentioned in Remark 2 (b). But this is obvious, since all the components of $\Lambda^{\ell-r+2}M$ contain a factor $\cong R$, while this is not the case for $S_{c_1}F_1^* \otimes \cdots \otimes S_{c_r}F_r^*$, $c_i \geq 1$ for every i. ∎

We end this section by pointing out that in view of the previous corollary, the study of $H_\cdot(\mathbf{K}_A)$ reduces to the study of $H_\cdot(\mathbf{K}'_A)$, where $\mathbf{K}'_A$ is defined as follows.

DEFINITION 7. For every A, $\mathbf{K}'_A$ is the complex

$$0 \to \Lambda^{\ell-r+1}M \to \Lambda^{\ell-r}M \to \ldots \to \Lambda^2 M \to M \to \overline{R},$$

obtained by a truncation of $\mathbf{K}_A$.

3. THE DETERMINANT OF $(\mathbf{K}_A)_{\underline{a}}$

In this section, we assume that the ground ring R is an algebraically closed field of characteristic 0.

It follows that in order to get information on $H_\cdot(\mathbf{K}'_A)$, we can resort to the Cayley–Koszul complexes $C^\cdot(m_1, \ldots, m_r; A)$ introduced (over $\mathbf{C}$) in [5, Sect. 2] (also cf. [6, Chapt. 14, Sect. 2]).

PROPOSITION 8. *Assume that $\underline{a}$ satisfies*

$$a_{i_0} - \sum_{i \neq i_0}(d_i - 1) \geq 1 \qquad \forall i_0 \in \{1, \ldots, r\}.$$

Then for every $v \leq \ell - r + 1$, it turns out that $(\Lambda^v M)_{\underline{a}} = C^p(m_1, \ldots, m_r; A)$, where

$$p = 1 - v + \sum_i (d_i - 1) \quad \text{and} \quad m_{i_0} = a_{i_0} - \sum_{i \neq i_0}(d_i - 1) \qquad \forall i_0 \in \{1, \ldots, r\}.$$

Proof. Since $v \leq \ell - r + 1$ implies $a_{i_0} + d_{i_0} - v \geq a_{i_0} - \sum_{i \neq i_0}(d_i - 1)$, the assumption rules out the possibility that $a_{i_0} + d_{i_0} - v = 0$. Hence Theorem 5 says that

$$(\Lambda^v M)_{\underline{a}} = \sum_{\alpha_1 + \cdots + \alpha_r = v,\, v-1,\, \alpha_i \neq d_i \ \forall i} L_{(d_1 - \alpha_1,\, 1^{a_1 + \alpha_1 - v})}F_1^* \otimes \cdots \otimes L_{(d_r - \alpha_r,\, 1^{a_r + \alpha_r - v})}F_r^*.$$

Setting both $p_{i_0} = d_{i_0} - \alpha_{i_0} - 1$ and $m_{i_0} = a_{i_0} - \sum_{i \neq i_0}(d_i - 1)$, for every $i_0 \in \{1, \dots, r\}$, the above summation becomes

$$\sum_{p_1+\cdots+p_r=p-1,\, p} L_{(p_1+1,\, 1^{p+m_1-p_1-1})}F_1^* \otimes \cdots \otimes L_{(p_r+1,\, 1^{p+m_r-p_r-1})}F_r^*.$$

But this is precisely the decomposition of $C^p(m_1, \dots, m_r; A)$ into G-irreducibles given in [6, Chapt. 14, Proposition 2.3]. (Beware of a misprint: in the sum $\bigoplus_{p_1,\dots,p_r} \dots$, each $S^{(p+m_i-p_i|p_i)}$ must be replaced by $S^{(p_i|p+m_i-p_i)}$.) ∎

Corollary 9. *Assume that* $\underline{a}$ *satisfies*

$$a_{i_0} - \sum_{i \neq i_0}(d_i - 1) \geq 1 \qquad \forall i_0 \in \{1, \dots, r\},$$

and set

$$m_{i_0} = a_{i_0} - \sum_{i \neq i_0}(d_i - 1) \qquad \forall i_0 \in \{1, \dots, r\}.$$

Then the Cayley–Koszul complex $C^{\cdot}(m_1, \dots, m_r; A)$ *is isomorphic to* $(\mathbf{K}'_A)_{\underline{a}}$.

Proof. By definition, cf. [5, Section 2], the boundary map of $C^{\cdot}(m_1, \dots, m_r; A)$ is exterior multiplication by the differential dA. As for $\partial^v_{\underline{A}}$, it is induced by the Koszul map $\Lambda^v \overline{E} \to \Lambda^{v-1}\overline{E}$, and by identifying $\Lambda^t \overline{E}$ with $\Lambda^{\ell-t}\overline{E}^*$, the latter becomes exterior multiplication by

$$\textstyle\sum_i (-1)^{i+1} \left(\sum_{j_i} \frac{\partial A}{\partial x^{(i)}_{j_i}} x^{(i)}_{j_i} \right).$$

∎

It is proven in [5, Theorem 2.1], that when $m_i \geq 0$ for every $i = 1, \dots, r$, the exactness of $C^{\cdot}(m_1, \dots, m_r; A)$ is equivalent to the condition $\mathrm{Det}(A) \neq 0$, where $\mathrm{Det}(A)$ stands for the hyperdeterminant of A. This fact, together with the previous corollary, establishes the following result.

Proposition 10. *If* $\mathrm{Det}(A) \neq 0$, *then* $(\mathbf{K}'_A)_{\underline{a}}$ *is exact for all multiindices* $\underline{a}$ *satisfying*

$$a_{i_0} - \sum_{i \neq i_0}(d_i - 1) \geq 1 \qquad \forall i_0 \in \{1, \dots, r\}.$$

Remark 3. The proposition above says that if $\mathrm{Det}(A) \neq 0$, then $H_{\cdot}(\mathbf{K}'_A)$ is concentrated in multidegrees $\underline{a}$ such that

$$a_{i_0} \leq \sum_{i \neq i_0}(d_i - 1) \qquad \text{for } \textit{some} \text{ index } i_0 \in \{1, \dots, r\}.$$

This sharply contrasts with the behavior of $H_{\ell-k}(\mathbf{K}_A)$, $k = 0, \ldots, r-2$, which is concentrated in multidegrees $\underline{a}$ such that

$$a_{i_0} > \sum_{i \neq i_0} (d_i - 1) \qquad \text{for } \textit{every} \text{ index } i_0 \in \{1, \ldots, r\}.$$

(For $v \geq \ell - r + 2$ implies

$$0 \leq a_{i_0} + \alpha_{i_0} - v \leq a_{i_0} + \alpha_{i_0} - \ell + r - 2$$

$$\leq a_{i_0} + d_{i_0} - \ell + r - 2 = a_{i_0} - \sum_{i \neq i_0} (d_i - 1) - 1 \quad .)$$

Finally, if $\mathrm{Det}(A) \neq 0$ and

$$a_{i_0} > \sum_{i \neq i_0} d_i \qquad \text{for } \textit{every} \text{ index } i_0 \in \{1, \ldots, r\},$$

then $(\mathbf{K}_A)_{\underline{a}}$ is exact (recall Corollary 6).

The following theorem summarizes most of the previous analysis. (For the notion of determinant of a complex, cf., e.g., [6, Appendix A].)

THEOREM 11. *Let the multiindex $\underline{a}$ satisfy the condition*

$$a_{i_0} > \sum_{i \neq i_0} d_i \qquad \forall i_0 \in \{1, \ldots, r\}.$$

Then the determinant of the complex $(\mathbf{K}_A)_{\underline{a}}$ is equal to $[\mathrm{Det}(A)]^{(-1)^{\ell-r+1}}$.

Proof. The modules $H_{\ell-k}(\mathbf{K}_A) = \Lambda^{\ell-k} M$, $k = 0, \ldots, r-2$, have no components in multidegree $\underline{a}$ by Corollary 6. Hence $\det(\mathbf{K}_A)_{\underline{a}} = \det(\mathbf{K}'_A)_{\underline{a}}$. But Corollary 9 says that $(\mathbf{K}'_A)_{\underline{a}}$ coincides with $C^{\cdot}(m_1, \ldots, m_r; A)$, where

$$m_{i_0} = a_{i_0} - \sum_{i \neq i_0} (d_i - 1) \qquad \forall i_0 \in \{1, \ldots, r\}.$$

By [5, Theorem 2.8], the determinant of $C^{\cdot}(m_1, \ldots, m_r; A)$ equals $[\mathrm{Det}(A)]^{(-1)^{\ell-r+1}}$, and we are done. ∎

A similar, but less precise statement holds for the multigraded components of the double complex $\mathbf{D}_A$. In order to give it, we make the following preparations.

For every $i = 1, \ldots, r$, we denote by $\mathbf{A}^{d_i}$ the affine space

$$\underbrace{R \times \cdots \times R}_{d_i},$$

and by $\overset{\circ}{\underline{x}}{}^{(i)}$ the point $(\overset{\circ}{x}{}_1^{(i)}, \ldots, \overset{\circ}{x}{}_{d_i}^{(i)}) \in \mathbf{A}^{d_i}$. $\underline{0}^{(i)}$ stands for the origin of $\mathbf{A}^{d_i}$.

PROPOSITION 12. *The $\overline{R}$-module M becomes free of rank $\ell - r + 1$ after localizing at every point $(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)}) \in \mathbf{A}^{d_1} \times \cdots \times \mathbf{A}^{d_r}$ such that at most one $\overset{\circ}{\underline{x}}{}^{(i)}$ is $\underline{0}^{(i)}$.*

Proof. For a point as in the statement, let $Q(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})$ denote the ideal of $\overline{R}$:

$$(x_1^{(1)} - \overset{\circ}{x}{}_1^{(1)}, \ldots, x_{d_1}^{(1)} - \overset{\circ}{x}{}_{d_1}^{(1)}; \ldots; x_1^{(r)} - \overset{\circ}{x}{}_1^{(r)}, \ldots, x_{d_r}^{(r)} - \overset{\circ}{x}{}_{d_r}^{(r)}).$$

Assume that $\overset{\circ}{\underline{x}}{}^{(i)} \neq \underline{0}^{(i)}$ for every i different from a fixed $i_0 \in \{1, \ldots, r\}$. When localizing at $Q(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})$, some indeterminate $x_{j_i}^{(i)}$ becomes invertible, for every $i \neq i_0$. Hence by a change of basis, we can assume that $\psi_{Q(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})}$ has the $(r-1) \times \ell$ matrix given in the proof of Proposition 2. It follows that the localization of M is a free $\overline{R}_{Q(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})}$-module of rank $\ell - r + 1$. ∎

COROLLARY 13. *For every $1 \leq i < j \leq r$, let X_{ij} denote*

$$\mathbf{A}^{d_1} \times \ldots \times \mathbf{A}^{d_{i-1}} \times \left\{\underline{0}^{(i)}\right\} \times \mathbf{A}^{d_{i+1}}$$
$$\times \ldots \times \mathbf{A}^{d_{j-1}} \times \left\{\underline{0}^{(j)}\right\} \times \mathbf{A}^{d_{j+1}} \times \cdots \times \mathbf{A}^{d_r}.$$

(a) *For every $v > 0$ and $u > 0$, the homology module $H_u(\mathbf{L}_{(v)}\psi)$ is supported in the union of all the sets X_{ij}.*

(b) *For every $k = 0, \ldots, r-2$, $\Lambda^{\ell-k}M$ is supported in the union of all the sets X_{ij}, and the Euler characteristic of $\mathbf{L}_{(\ell-k)}\psi$ is zero.*

Proof. Since after localization M is free of rank $\ell - r + 1$, ψ is a split monomorphism and $\mathbf{L}_{(v)}\psi$ is acyclic (this proves (a)). Moreover all exterior powers $\Lambda^v M$, $v \geq \ell - r + 2$, must vanish (and (b) follows). ∎

Remark 4. (a) The corollary above confirms that $\partial_A^{\ell-k} = 0$ for every $k = 0, \ldots, r-2$. (Cf. Corollary 6.)

(b) The corollary above is consistent with Proposition 4 (b), for the indicated localization obviously kills all quotients $I_k(U)/I_{k+1}(U)$. The new information is that also every $\operatorname{Ker}\left\{\Lambda^{\ell-k}M \to I_k(U)/I_{k+1}(U)\right\}$ gets killed.

THEOREM 14. *There exist positive integers $n_1, \ldots, n_r$ such that for every multiindex $\underline{a}$ satisfying*

$$a_{i_0} \geq n_{i_0} \qquad \forall i_0 \in \{1, \ldots, r\},$$

the determinant of the double complex $(\mathbf{D}_A)_{\underline{a}}$ is equal to $[\operatorname{Det}(A)]^{(-1)^{\ell-r+1}}$.

Proof. The columns of $(\mathbf{D}_A)_{\underline{a}}$ are the multigraded components of the Schur complexes $\mathbf{L}_{(v)}\psi$. Corollary 13 says that for every positive v, the homology modules $H_u(\mathbf{L}_{(v)}\psi)$, $u > 0$, are supported in the union of the sets X_{ij}. But those are finitely generated multigraded modules, hence there exist positive integers $n_1, \ldots, n_r$ such that $a_{i_0} \geq n_{i_0}$ $\forall i_0$ implies

$$(H_u(\mathbf{L}_{(v)}\psi))_{\underline{a}} = 0 \qquad \forall u > 0 \quad \forall v > 0.$$

Thus the multigraded component $(\mathbf{L}_{(v)}\psi)_{\underline{a}}$ is a resolution of the multigraded component $(\Lambda^v M)_{\underline{a}}$. This means that the determinant of $(\mathbf{D}_A)_{\underline{a}}$ equals the determinant of $(\overline{\mathbf{K}}_A)_{\underline{a}}$. Since we can obviously choose $n_1, \ldots, n_r$ so that the condition $a_{i_0} > \sum_{i \neq i_0} d_i$ $\forall i_0$ is satisfied, Theorem 11 concludes the proof. ∎

Remark 5. It would be interesting to find the integers $n_1, \ldots, n_r$ above in an explicit way.

The rest of this section deals with the ideal $\mathrm{Im}(\varphi_A) = \mathrm{Im}(\overline{\varphi}_A)$, which we denote by J_A (J stands for Jacobian). Explicitly,

$$J_A = \left(\frac{\partial A}{\partial x_1^{(1)}}, \ldots, \frac{\partial A}{\partial x_{d_1}^{(1)}}; \ldots; \frac{\partial A}{\partial x_1^{(r)}}, \ldots, \frac{\partial A}{\partial x_{d_r}^{(r)}} \right).$$

(Some special instances of this ideal are studied in [3].)

PROPOSITION 15. *If* $\mathrm{Det}(A) \neq 0$, *then* $\mathrm{depth}(J_A)_Q \geq \ell - r + 1$ *for every maximal ideal Q of $\overline{R}$, which does not include the ideal generated by all products of type* $x_{j_1}^{(1)} \cdot \ldots \cdot x_{j_r}^{(r)}$.

Proof. It suffices to show that $\mathrm{Det}(A) = 0$, if there exists a maximal ideal Q not including $(x_{j_1}^{(1)} \cdot \ldots \cdot x_{j_r}^{(r)})$ and such that $\mathrm{depth}(J_A)_Q \leq \ell - r$. It is easy to see (cf. e.g., [5, Sect. 1]) that $\mathrm{Det}(A) = 0$ if, and only if, all the partial derivatives $\partial A / \partial x_{j_i}^{(i)}$, $1 \leq i \leq r$, $1 \leq j_i \leq d_i$, vanish on some point $(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})$ of $\mathbf{A}^{d_1} \times \cdots \times \mathbf{A}^{d_r}$ such that $\overset{\circ}{\underline{x}}{}^{(i)} \neq \underline{0}^{(i)}$ for every i. Hence it is enough to prove that if there exists a maximal ideal Q not including $(x_{j_1}^{(1)} \cdot \ldots \cdot x_{j_r}^{(r)})$ and such that $\mathrm{depth}(J_A)_Q \leq \ell - r$, then all $\partial A / \partial x_{j_i}^{(i)}$, $1 \leq i \leq r$, $1 \leq j_i \leq d_i$, vanish on some $(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})$ such that $\overset{\circ}{\underline{x}}{}^{(i)} \neq \underline{0}^{(i)}$ for every i.

Now, Q must include J_A (otherwise, $\mathrm{depth}(J_A)_Q$ would not be $\leq \ell - r$). Say (notation as in Proposition 12), $Q = Q(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})$, for some $(\overset{\circ}{\underline{x}}{}^{(1)}, \ldots, \overset{\circ}{\underline{x}}{}^{(r)})$ on which all $\partial A / \partial x_{j_i}^{(i)}$ vanish. Since Q does not include $(x_{j_1}^{(1)} \cdot \ldots \cdot x_{j_r}^{(r)})$, it follows that for every $i = 1, \ldots, r$, there exists an indeterminate $x_{j_i}^{(i)}$ not belonging to Q. Hence $\overset{\circ}{\underline{x}}{}^{(i)} \neq \underline{0}^{(i)}$ for every i, as required. ∎

COROLLARY 16. *If* $\mathrm{Det}(A) \neq 0$, *then* $\mathbf{K}_A \otimes_{\overline{R}} \overline{R}_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})}$ *is acyclic for each point* $(\overset{\circ}{\underline{x}}^{(1)}, \ldots, \overset{\circ}{\underline{x}}^{(r)})$ *such that* $\overset{\circ}{\underline{x}}^{(i)} \neq \underline{0}^{(i)}$ *for every* i.

Proof. Since $\overset{\circ}{\underline{x}}^{(i)} \neq \underline{0}^{(i)}$ implies $x_{j_i}^{(i)} \notin Q(\overset{\circ}{\underline{x}}^{(1)}, \ldots, \overset{\circ}{\underline{x}}^{(r)})$ for a suitable j_i, $Q(\overset{\circ}{\underline{x}}^{(1)}, \ldots, \overset{\circ}{\underline{x}}^{(r)})$ does not contain a product $x_{j_1}^{(1)} \cdot \ldots \cdot x_{j_r}^{(r)}$. Hence the previous proposition applies, and

$$\mathrm{depth}(J_A)_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})} \geq \ell - r + 1.$$

Lemma 17 below shows that $(J_A)_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})}$ is generated by $\ell - r + 1$ partial derivatives. Therefore:

$$\mathrm{depth}(J_A)_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})} = \ell - r + 1,$$

and the partial derivatives generating $(J_A)_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})}$ form a regular sequence (cf., e.g., [7, Theorem 129]). Since $M_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})}$ is free of rank $\ell - r + 1$, by Proposition 12, it follows that $(\mathbf{K}_A)_{Q(\overset{\circ}{\underline{x}}^{(1)},\ldots,\overset{\circ}{\underline{x}}^{(r)})}$ is acyclic.

Modulo Lemma 17, the proof is complete. ∎

LEMMA 17. *After localizing at any point* $(\overset{\circ}{\underline{x}}^{(1)}, \ldots, \overset{\circ}{\underline{x}}^{(r)})$ *such that at most one* $\overset{\circ}{\underline{x}}^{(i)}$ *is* $\underline{0}^{(i)}$, J_A *is generated by* $\ell - r + 1$ *partial derivatives*.

Proof. As in the proof of Proposition 12, assume that $\overset{\circ}{\underline{x}}^{(i)} \neq \underline{0}^{(i)}$ for every i different from a fixed $i_0 \in \{1, \ldots, r\}$. Hence for every $i \neq i_0$, there exists an indeterminate $x_{j_i}^{(i)}$ which becomes invertible after localizing. But then

$$\sum_{h_i=1}^{d_i} x_{h_i}^{(i)} \frac{\partial A}{\partial x_{h_i}^{(i)}} = A = \sum_{h_{i_0}=1}^{d_{i_0}} x_{h_{i_0}}^{(i_0)} \frac{\partial A}{\partial x_{h_{i_0}}^{(i_0)}}$$

yields that the corresponding $\partial A / \partial x_{j_i}^{(i)}$ is a linear combination of the other partial derivatives. This lowers the number of generators to $\ell - (r-1) = \ell - r + 1$. ∎

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